

A DEGREE MAP ON UNIMODULAR ROWS

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ABSTRACT. Let k be a field and let $(\mathbb{A}^n - 0)$ be the punctured affine space. We associate to any morphism $g : (\mathbb{A}^n - 0) \rightarrow (\mathbb{A}^n - 0)$ an element in the Witt group $W(k)$ that we call the degree of g . We then use this degree map to give a negative answer to a question of M. V. Nori about unimodular rows.

INTRODUCTION

Let R be a ring and let P be a projective R -module such that $P \oplus R \simeq R^{n+1}$ for some $n \in \mathbb{N}$. The obvious question is to know if such an isomorphism yields an isomorphism $P \simeq R^n$. In general, this is not the case and finding general conditions for the answer to be positive is an important and extensively studied problem. Recall that one can associate to a projective module as above a unimodular row of length $n+1$, i.e. a row (a_0, \dots, a_n) such that $\sum Ra_i = R$. Let $Um_{n+1}(R)$ be the set of unimodular rows of length $n+1$. It is clear that $GL_{n+1}(R)$ acts on $Um_{n+1}(R)$ by right multiplication, and therefore so does any subgroup of $GL_{n+1}(R)$. In general, one is particularly interested by the groups $E_{n+1}(R)$ generated by elementary matrices and $SL_{n+1}(R)$, because $Um_{n+1}(R)/E_{n+1}(R)$ corresponds to some cohomotopy group in the topological situation and then carries a group structure when n is "reasonable" compared to the Krull dimension of d ([19, Theorem 4.1]) and because $Um_{n+1}(R)/SL_{n+1}(R)$ classifies up to isomorphism projective modules P such that $P \oplus R \simeq R^{n+1}$. One says that a unimodular row v is completable if $v \in e_1 SL_{n+1}(R)$ where $e_1 = (1, 0, \dots, 0)$. Thus the question to know if $P \oplus R \simeq R^{n+1}$ implies $P \simeq R^n$ reduces to the question to know if the associated unimodular row is completable.

The first general condition on unimodular rows to be completable was given by Suslin ([18, Theorem 2]).

Theorem. *Let R be an arbitrary ring, let (a_0, \dots, a_n) be a unimodular row and let m_0, \dots, m_n be positive integers. Suppose that $\prod_{i=0}^n m_i$ is divisible by $n!$. Then the row $(a_0^{m_0}, \dots, a_n^{m_n})$ is completable.*

If R is an algebra over a field k , observe that an element $v \in Um_{n+1}(R)$ corresponds naturally to a morphism of schemes $f : \text{Spec}(R) \rightarrow (\mathbb{A}_k^{n+1} - 0)$. Now the homomorphism $\phi : k[x_0, \dots, x_n] \rightarrow k[x_0, \dots, x_n]$ defined by $\phi(x_i) = x_i^{m_i}$ induces a morphism $\varphi : (\mathbb{A}_k^{n+1} - 0) \rightarrow (\mathbb{A}_k^{n+1} - 0)$ and Suslin's result reads as follows: Let $v \in Um_{n+1}(R)$ corresponding to the morphism $f : \text{Spec}(R) \rightarrow (\mathbb{A}_k^{n+1} - 0)$. Then the row w corresponding to φf is completable. In view of this, M. V. Nori asked the following very natural question ([14]):

Question (M. V. Nori). *Let $R = k[x_0, \dots, x_n]$ be the polynomial ring in $n+1$ variables, where k is a field. Let $\phi : R \rightarrow A$ be a k -algebra homomorphism such that $\sum \phi(x_i)A = A$. Let f_0, \dots, f_n such that $\text{rad}(f_0, \dots, f_n) = (x_0, \dots, x_n)$. Assume*

$l(R/(f_0, \dots, f_n))$ is divisible by $n!$. Then is the unimodular row $(\phi(f_0), \dots, \phi(f_n))$ completable?

Mohan Kumar proved that this question has a positive answer when the base field is algebraically closed and the polynomials f_i are homogeneous ([14]). The purpose of this article is to show that the answer to Nori's question is negative in general for n odd. More precisely, we prove the following theorem (Theorem 4.7 in the text).

Theorem. *Let k be a field such that $\sqrt{-1} \notin k$. Consider the unimodular row $(x_1, x_2, x_3) \in Um_3(S_3)$ and the map $g : (\mathbb{A}^3 - 0) \rightarrow (\mathbb{A}^3 - 0)$ defined by the algebra homomorphism $\varphi : k[x_1, x_2, x_3] \rightarrow k[x_1, x_2, x_3]$ given by $\varphi(x_1) = x_1^2 - x_2^2$, $\varphi(x_2) = x_1x_2$ and $\varphi(x_3) = x_3$. Then $k[x_1, x_2, x_3]/(x_1^2 - x_2^2, x_1x_2, x_3)$ is of length 4, but $(x_1^2 - x_2^2, x_1x_2, x_3) \in Um_3(S_3)$ is not completable.*

Our guess is that Nori's question has an affirmative answer when n is even. We now describe our strategy to prove this result. Consider the affine k -algebra $A_n = k[x_1, \dots, x_n, y_1, \dots, y_n]/(\sum x_i y_i - 1)$ and set $S_n = \text{Spec}(A_n)$. There is a natural projection $p : S_n \rightarrow (\mathbb{A}_k^n - 0)$ with affine fibres. Now it is not hard to see that any unimodular row $f : \text{Spec}(R) \rightarrow (\mathbb{A}_k^n - 0)$ factors through S_n , i.e. there exists a morphism $f' : \text{Spec}(R) \rightarrow S_n$ such that $f = pf'$. This is the reason why we call S_n the unimodular affine scheme. To prove a general property of the unimodular rows of length n over any k -algebra R , it suffices then to prove it over S_n .

Suppose for a while that $k = \mathbb{R}$. Let $f : S_n \rightarrow (\mathbb{A}_{\mathbb{R}}^n - 0)$ be any morphism. Considering only the real closed points, and using the fact that the projection $p : S_n \rightarrow (\mathbb{A}_k^n - 0)$ has affine fibers, we are left up to homotopy with a polynomial map $f : (\mathbb{R}^n - 0) \rightarrow (\mathbb{R}^n - 0)$. We can consider its (Brouwer) degree, defined as the sum of the signs of the Jacobian of f at all preimages (under f) near 0 of a regular value of f near 0. Since the Witt group of \mathbb{R} is equal to \mathbb{Z} , we can see the degree as an element of $W(\mathbb{R})$. It is worth noting that the degree of two homotopic maps are the same, and that the degree of the constant map is 0. Thus a map with a non constant degree cannot be homotopic to a constant map. Translating this rather easy topological situation to an algebraic situation will be our motivation for the rest of the paper.

We begin with the definition of a symbol $\phi : Um_n(R) \rightarrow GW_{red}^{n-1}(R)$, where the latter is a quotient of the derived Grothendieck-Witt groups of R , as defined by Walter ([22], see also [2]). This definition requires some knowledge of Grothendieck-Witt groups, and we therefore spend a few sections to remind some basic facts about these groups. The group $GW_{red}^{n-1}(R)$ being homotopy invariant, this symbol factors through the action of the group $E_n(R)$ generated by elementary matrices to give a symbol $\phi : Um_n(R)/E_n(R) \rightarrow GW_{red}^{n-1}(R)$ for $n \geq 3$.

If n is odd, the symbol yields a symbol $\Phi : Um_n(R)/SL_n(R) \rightarrow GW_{red}^{n-1}(R)$ while this is not the case for n even. This rather odd fact will probably be explained when the computation of the Grothendieck-Witt groups of SL_n will be fully achieved (we expect a different answer depending on the parity of n). We then compute the Grothendieck-Witt groups of S_n to get $GW_{red}^{n-1}(S_n) = W(k)$, the Witt group of the base field. Under this identification $\Phi(p) = \langle 1 \rangle$, where $p : S_n \rightarrow (\mathbb{A}_k^n - 0)$ is the projection.

In Section 4, we associate to any morphism $f : (\mathbb{A}_k^n - 0) \rightarrow (\mathbb{A}_k^n - 0)$ a degree generalizing the (Brouwer) degree. This degree map take coefficient in the Witt group $W(k)$ of the base field k , and we ingeniously denote by $\deg(f)$ the degree of the map f . The basic fact here is that $\Phi(f \circ p) = \deg(f)$ in $W(k)$! This is exactly the obstruction we need to exhibit our counter-example to Nori's question.

In view of the example provided by Theorem 4.7, we propose a strengthened conjecture (Question 4.8 in the text) which is equivalent to the original conjecture when the base field k is algebraically closed.

Question. *Let R be a k -algebra, $n \in \mathbb{N}$ be odd and let $f : \text{Spec}(R) \rightarrow (\mathbb{A}^n - 0)$ be a unimodular row. Let $\varphi : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ such that $\varphi(\mathfrak{m}) \subset \mathfrak{m}$ and that $(n-1)!$ divides the length of $k[x_1, \dots, x_n]/\varphi(\mathfrak{m})$. Let $g : (\mathbb{A}^n - 0) \rightarrow (\mathbb{A}^n - 0)$ be the morphism induced by φ . If the degree $\deg(g) = 0$, then the unimodular row $gf : \text{Spec}(R) \rightarrow (\mathbb{A}^n - 0)$ is completable.*

Conventions. Any scheme is of finite type and separated over a field of characteristic different from 2.

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1. UNIMODULAR ROWS

Let R be a k -algebra. A unimodular row of length $n \geq 2$ is a surjective homomorphism $R^n \rightarrow R$. We denote by $Um_n(R)$ the set of unimodular rows of length n . The group $GL_n(R)$ acts on $Um_n(R)$ by left composition, and so do any subgroup. We say that $v \in Um_n(R)$ is completable if it is the first row of a matrix in $SL_n(R)$.

Observe that $Um_n(R) = \text{Hom}(\text{Spec}(R), \mathbb{A}^n - 0)$. Let $r : SL_n \rightarrow (\mathbb{A}^n - 0)$ be the projection on the first row. In this setting, a unimodular row $f : \text{Spec}(R) \rightarrow (\mathbb{A}^n - 0)$ is completable if f factors through SL_n , i.e if there exists a morphism $f' : \text{Spec}(R) \rightarrow SL_n$ such that the diagram

$$\begin{array}{ccc} \text{Spec}(R) & \xrightarrow{f'} & SL_n \\ & \searrow f & \downarrow r \\ & & (\mathbb{A}^n - 0) \end{array}$$

commutes.

1.1. Steinberg symbols. Let R be a ring. A Steinberg symbol is a pair (ρ, G) , where G is a group and $\rho : Um_n(R) \rightarrow G$ is a map such that the following relations hold:

- (1) $\rho(v) = \rho(vE)$ for any $E \in E_n(R)$.
- (2) $\rho(x, v_2, \dots, v_n)\rho(1-x, v_2, \dots, v_n) = \rho(x(1-x), v_2, \dots, v_n)$ if (x, v_2, \dots, v_n) and $(1-x, v_2, \dots, v_n)$ are unimodular.

It is clear that a universal Steinberg symbol exists. We denote it by $(St_n(R), st)$. Observe that by definition a Steinberg symbol $\rho : Um_n(R) \rightarrow G$ induces a map $\rho : Um_n(R)/E_n(R) \rightarrow G$. If R is a ring of Krull dimension d with $2 \leq d \leq 2n - 4$, then W. van der Kallen proved that the universal Steinberg symbol $(St_n(R), st)$ induces a bijection $st : Um_n(R)/E_n(R) \rightarrow St_n(R)$ ([20, Theorem 3.3]). Moreover, $St_n(R)$ is abelian in this situation ([20, Remark 3.4]), and therefore st endows $Um_n(R)/E_n(R)$ with the structure of an abelian group.

1.2. Unimodular rows and projective modules. For any unimodular element $a = (a_1, \dots, a_n) \in Um_n(R)$ we denote by $P(a)$ the cokernel of the homomorphism $a^t : R \rightarrow R^n$. We have then an exact sequence

$$0 \longrightarrow R \xrightarrow{a^t} R^n \longrightarrow P(a) \longrightarrow 0.$$

Usually, one defines $P(a)$ to be the kernel of $a : R^n \rightarrow R$ and it is clear that our definition is the dual of this one. If n is odd, then $s(a) : R^n \rightarrow R$ given by the row $(-a_2, a_1, \dots, -a_{n-1}, a_{n-2}, 0)$ has the property that $s(a)(a^t) = 0$. It induces a homomorphism $P(a) \rightarrow R$ that we still denote $s(a)$. If n is even, then we can define $s(a)$ by the row $(-a_2, a_1, \dots, -a_n, a_{n-1})$ giving a surjective homomorphism $s(a) : P(a) \rightarrow R$.

The sequence

$$0 \longrightarrow R \xrightarrow{a^t} R^n \longrightarrow P(a) \longrightarrow 0$$

induces an isomorphism $\chi(a) : \det P(a) \simeq R \cdot e_1 \wedge \dots \wedge e_n$, where e_1, \dots, e_n is the usual basis of R^n . If $b = (b_1, \dots, b_n)$ is such that $ba^t = 1$ and f_i is the image of e_i in $P(a)$ for $1 \leq i \leq n$, a straightforward computation shows that $\chi(a)^{-1}(e_1 \wedge \dots \wedge e_n) = \sum_{i=1}^n (-1)^i b_i \cdot f_1 \wedge \dots \wedge f_{i-1} \wedge f_{i+1} \wedge \dots \wedge f_n$. We denote by $\omega(a)$ this generator of $\det P(a)$. Since $a_i \omega(a) = (-1)^i f_1 \wedge \dots \wedge f_{i-1} \wedge f_{i+1} \wedge \dots \wedge f_n$ for all $1 \leq i \leq n$, we see that $\omega(a)$ is independent of the choice of b . Indeed, if c is such that $ca^t = 1$, then

$$\omega(a) = \left(\sum_{i=1}^n c_i a_i \right) \omega(a) = \sum_{i=1}^n (-1)^i c_i f_1 \wedge \dots \wedge f_{i-1} \wedge f_{i+1} \wedge \dots \wedge f_n.$$

2. GROTHENDIECK-WITT GROUPS

2.1. The basics. Let X be a scheme over $\text{Spec}(k)$. For any $j \in \mathbb{Z}$ and any line bundle L over X , we denote by $GW^j(X, L)$ the j -th Grothendieck-Witt group of the derived category of bounded complexes of coherent locally free \mathcal{O}_X -modules, with duality induced by the functor $\text{Hom}_{\mathcal{O}_X}(_, L)$ ([22, §2]). We denote by $W^j(X, L)$ the Witt groups of this category ([2, Definition 1.4.3]). By definition, there is an exact sequence ([22, Theorem 2.6])

$$K_0(X) \longrightarrow GW^j(X, L) \longrightarrow W^j(X, L) \longrightarrow 0.$$

The Grothendieck-Witt groups are contravariant: If $f : Y \rightarrow X$ is a morphism of schemes, then there are homomorphisms $f^* : GW^j(X, L) \rightarrow GW^j(Y, f^*L)$. In particular, the structural morphism $p : X \rightarrow \text{Spec}(k)$ induces a homomorphism $p^* : GW^j(k) \rightarrow GW^j(X)$ for any $j \in \mathbb{Z}$. We denote by $GW_{red}^j(X)$ its cokernel (which is a direct factor of $GW^j(X)$ if X has a rational point).

2.2. Transfers. Suppose that $i : Y \rightarrow X$ is a closed immersion. Then we can consider the category of bounded complexes of coherent locally free \mathcal{O}_X -modules whose homology is supported on Y . We will denote its Grothendieck-Witt groups by $GW_Y^j(X, L)$. We can sometimes identify the groups with support on Y with the groups associated to Y . This procedure is usually called *dévissage*. In the special situation where X and Y are smooth, the *dévissage* can be described as follows: Let $\bar{i} : (Y, \mathcal{O}_Y) \rightarrow (X, i_*\mathcal{O}_Y)$ be the morphism of ringed spaces induced by i . Suppose that i is of pure codimension d . If L is a line bundle over X , then $N := \bar{i}^* \text{Ext}_{\mathcal{O}_X}^d(i_*\mathcal{O}_Y, i^*L)$ is a line bundle over Y and there is a transfer morphism ([7, Theorem 6.2])

$$i_* : GW^j(Y, N) \rightarrow GW_Y^{j+d}(X, L).$$

At the level of the Witt groups, this homomorphism is in fact an isomorphism ([13, Theorem 3.2]). This fact is also true for Grothendieck-Witt groups, but we don't use it here. In case N is trivial, i_* becomes an isomorphism

$$i_* : GW^j(Y) \rightarrow GW_Y^{j+d}(X, L).$$

Observe that i_* now depends on a trivialization isomorphism $\mathcal{O}_Y \simeq N$.

2.3. Finite length modules and *dévissage*. Let R be a regular local ring of dimension d . Let $\mathcal{M}_{fl}(R)$ be the (abelian) category of finite length R -modules. The functor $M \mapsto \widehat{M} := \text{Ext}_R^d(M, R)$ endows $\mathcal{M}_{fl}(R)$ with a duality with canonical isomorphism defined as follows: Let $\pi : \mathcal{P} \rightarrow M$ be a projective resolution of M . The canonical identification of \mathcal{P} with its bidual $\mathcal{P}^{\vee\vee}$ yields an isomorphism $\eta : M \rightarrow \widehat{\widehat{M}}$ which is independent of the projective resolution ([9, Theorem 3.3.2] for instance). We set $\varpi := (-1)^{d(d-1)/2}\eta$. Having a duality functor and a canonical isomorphism, we can define the Witt group of finite length modules $W^{fl}(R)$ following [15].

Suppose that X is smooth of dimension d and that $x \in X$ is a closed point. We consider the Witt group $W_x^d(X)$. By the above section, we know that there is an isomorphism

$$i_* : W(k(x), \text{Ext}_{\mathcal{O}_X}^d(k(x), \mathcal{O}_X)) \rightarrow W_x^d(X).$$

We can also interpret this isomorphism using finite length $\mathcal{O}_{X,x}$ -modules following [4, §6].

Observe first that localizing at x induces an isomorphism $W_x^j(X) \rightarrow W_x^j(\mathcal{O}_{X,x})$ for any $j \in \mathbb{N}$. Now $W^{fl}(\mathcal{O}_{X,x}) \simeq W_x^d(\mathcal{O}_{X,x})$ by [4, Theorem 6.1, Proposition 6.2]. The map is defined by sending a finite length module (endowed with a form ϕ) to a projective resolution of this module (endowed with a form induced by ϕ). Finally, there is a canonical isomorphism $W(k(x), \text{Ext}_{\mathcal{O}_X}^d(k(x), \mathcal{O}_X)) \rightarrow W^{fl}(\mathcal{O}_{X,x})$ ([9, Appendix E.2]). The isomorphism i_* is the composition

$$W(k(x), \text{Ext}_{\mathcal{O}_X}^d(k(x), \mathcal{O}_X)) \twoheadrightarrow W^{fl}(\mathcal{O}_{X,x}) \twoheadrightarrow W_x^d(\mathcal{O}_{X,x}) \twoheadrightarrow W_x^d(X).$$

2.4. Euler classes. Let X be a scheme and \mathcal{E} be a rank n coherent locally free \mathcal{O}_X -module over X . Let $s : \mathcal{E} \rightarrow \mathcal{O}_X$ be any section (possibly trivial). Then we can consider the Koszul complex $Kos(s)$ associated to s . For any $1 \leq i \leq n$, we have isomorphisms $\varphi_i : \wedge^i \mathcal{E} \rightarrow \text{Hom}_{\mathcal{O}_X}(\wedge^{n-i} \mathcal{E}, \det \mathcal{E})$ defined by $\varphi_i(p)(q) = p \wedge q$. Let

$$\rho(s) : Kos(s) \rightarrow T^n \text{Hom}_{\mathcal{O}_X}(Kos(s), \det \mathcal{E})$$

be the isomorphism defined in degree i by $\rho(s)_i = (-1)^{in+i(i-1)/2+n(n-1)/2}\varphi_i$ (note that the ρ_i given in [3, Remark 4.2] do not define a morphism of complexes). It

turns out that $\rho(s)$ is symmetric for the n -th shifted duality, and therefore it defines an element in $GW^n(X, \det \mathcal{E})$.

Lemma 2.1. *Suppose that X is regular. let $s : \mathcal{E} \rightarrow \mathcal{O}_X$ and $s' : \mathcal{E} \rightarrow \mathcal{O}_X$ be two sections. Then $\rho(s) = \rho(s')$ in $GW^n(X, \det \mathcal{E})$.*

Proof. Let $p : X \times \mathbb{A}^1 \rightarrow X$ be the projection. We consider the section

$$(tp^*s + (1-t)p^*s') : p^*\mathcal{E} \rightarrow \mathcal{O}_{X \times \mathbb{A}^1}.$$

Since X is regular, $p^* : GW^n(X, \det \mathcal{E}) \rightarrow GW^n(X \times \mathbb{A}^1, p^*\det \mathcal{E})$ is an isomorphism ([10, Proposition 1.1]). There exists then $\alpha \in GW^n(X, \det \mathcal{E})$ such that $p^*\alpha = \rho(tp^*s + (1-t)p^*s')$. Evaluating at $t = 0$ and $t = 1$, we get $\alpha = \rho(s) = \rho(s')$. \square

Definition 2.2. We call Euler class of \mathcal{E} the element $\rho(s)$ in $GW^n(X, \det \mathcal{E})$ for any section $s : \mathcal{E} \rightarrow \mathcal{O}_X$. We denote it by $e(\mathcal{E})$.

If $f : Y \rightarrow X$ is a morphism of regular schemes, then it is clear from the definition that $e(f^*\mathcal{E}) = f^*e(\mathcal{E})$.

Let $g : \mathcal{E} \rightarrow \mathcal{E}$ be an isomorphism such that $\det g = 1$ and let $p : E \rightarrow X$ be the total space of \mathcal{E} . If we also denote by $g : E \rightarrow E$ the morphism induced by g , then we see from the definition of the Euler class that $g^*e(p^*\mathcal{E}) = e(p^*\mathcal{E})$. It follows that the Euler class is invariant under the action of $SL(\mathcal{E})$.

Suppose next that $\chi : \det \mathcal{E} \rightarrow \mathcal{O}_X$ is an isomorphism. It induces an isomorphism $GW^n(X, \det \mathcal{E}) \rightarrow GW^n(X)$. We denote by $e(\mathcal{E}, \chi)$ the image of $e(\mathcal{E})$ under this isomorphism. It depends of course on the isomorphism χ , but it is still invariant under the action of $SL(\mathcal{E})$.

2.5. Some computations. Let X be a scheme. If $x \in \mathcal{O}_X(X)$ is a global section, we denote by $Z(x)$ the vanishing locus of x , which is a closed subset in X (since X is noetherian by our conventions).

Let $(a_1, \dots, a_n) \in \mathcal{O}_X(X)$ and let $Kos(a_1, \dots, a_n)$ be the Koszul complex associated to the (non necessarily regular) sequence of global sections (a_1, \dots, a_n) . As in the previous section, this complex is endowed with a symmetric isomorphism

$$\rho(a_1, \dots, a_n) : Kos(a_1, \dots, a_n) \rightarrow T^n Kos(a_1, \dots, a_n)^\vee.$$

Let $Z := \cap_{i=1}^n Z(a_i)$ be the closed subset of X where the sections all vanish. Since the homology of $Kos(a_1, \dots, a_n)$ is concentrated on Z , the above symmetric isomorphism defines an element in $GW_Z^n(X)$ (and $W_Z^n(X)$) that we still denote by $\rho(a_1, \dots, a_n)$.

We can also consider the form $\rho(a_1, \dots, a_{n-1})$ on $Kos(a_1, \dots, a_{n-1})$ whose homology is supported on $Z' := \cap_{i=1}^{n-1} Z(a_i)$. If $U = X - Z$, then

$$(Z(a_n) \cap U) \cap (Z' \cap U) = \emptyset.$$

It follows that

$$a_n \rho(a_1, \dots, a_{n-1}) : Kos(a_1, \dots, a_{n-1}) \rightarrow T^n Kos(a_1, \dots, a_{n-1})^\vee$$

is an isomorphism on U , and we denote its class in $GW^{n-1}(U)$ by $\theta(a_1, \dots, a_{n-1})$. The following result is a straightforward consequence of [3, Remark 4.2] and the Leibnitz formula for symmetric isomorphisms ([1, Theorem 5.2]):

Lemma 2.3. *Let $\partial : GW^{n-1}(U) \rightarrow W_Z^n(X)$ be the connecting homomorphism in the localization sequence associated to $U \subset X$. Then $\partial(\theta(a_1, \dots, a_{n-1})) = \rho(a_1, \dots, a_n)$.*

Lemma 2.4. *For any $n \in \mathbb{N}$, we have $GW_{red}^{n-1}(\mathbb{A}^n - 0) = W(k) \cdot \theta(x_1, \dots, x_n)$.*

Proof. The long exact sequence of localization reads as

$$\dots \rightarrow GW_{\{0\}}^{n-1}(\mathbb{A}^n) \rightarrow GW^{n-1}(\mathbb{A}^n) \rightarrow GW^{n-1}(\mathbb{A}^n - 0) \xrightarrow{\partial} W_{\{0\}}^n(\mathbb{A}^n) \rightarrow \dots$$

By homotopy invariance of Grothendieck-Witt groups ([10, Proposition 1.1]), we see that $GW^{n-1}(\mathbb{A}^n) = GW^{n-1}(k)$. Using this, we see that the choice of a rational point in $\mathbb{A}^n - 0$ gives a splitting of the map $GW^{n-1}(\mathbb{A}^n) \rightarrow GW^{n-1}(\mathbb{A}^n - 0)$. We then get a split exact sequence

$$0 \longrightarrow GW^{n-1}(k) \longrightarrow GW^{n-1}(\mathbb{A}^n - 0) \xrightarrow{\partial} W_{\{0\}}^n(\mathbb{A}^n) \longrightarrow 0$$

showing that $GW_{red}^{n-1}(\mathbb{A}^n - 0) \simeq W_{\{0\}}^n(\mathbb{A}^n)$.

By dévissage, $W_{\{0\}}^n(\mathbb{A}^n) = W(k) \cdot \rho(x_1, \dots, x_n)$. We can use Lemma 2.3 to conclude. \square

Remark 2.5. Let $R = k[x_1, \dots, x_n]$ and $\mathfrak{m} = (x_1, \dots, x_n)$. Observe that the above dévissage isomorphism $W_{\{0\}}^n(\mathbb{A}^n) \simeq W(k)$ amounts to choose a generator of $\text{Ext}_R^n(R/\mathfrak{m}, R)$. When we write $W_{\{0\}}^n(\mathbb{A}^n) = W(k) \cdot \rho(x_1, \dots, x_n)$, we implicitly choose the Koszul complex associated to the sequence (x_1, \dots, x_n) as a generator.

Remark 2.6. The action of $W(k)$ on $\theta(x_1, \dots, x_n)$ reads as follows: If $\alpha \in k^\times$, then $\langle \alpha \rangle \cdot \theta(x_1, \dots, x_n) = \theta(x_1, \dots, x_{n-1}, \alpha x_n)$.

We will need the following lemma in the next section:

Lemma 2.7. *For any $n \geq 1$, Let $j : \mathbb{A}^{n-1} \rightarrow \mathbb{A}^n$ be the inclusion obtained by setting $x_n = 0$. Let $j' : (\mathbb{A}^{n-1} - 0) \rightarrow (\mathbb{A}^n - 0)$ be the induced closed immersion. Then the transfer homomorphism $(j')_* : GW^{n-2}(\mathbb{A}^{n-1} - 0) \rightarrow GW^{n-1}(\mathbb{A}^n - 0)$ induces an isomorphism*

$$(j')_* : GW_{red}^{n-2}(\mathbb{A}^{n-1} - 0) \rightarrow GW_{red}^{n-1}(\mathbb{A}^n - 0)$$

Proof. The transfer homomorphism associated to j induces a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & GW^{n-2}(\mathbb{A}^{n-1}) & \longrightarrow & GW^{n-2}(\mathbb{A}^{n-1} - 0) & \xrightarrow{\partial} & W_{\{0\}}^{n-1}(\mathbb{A}^{n-1}) \longrightarrow 0 \\ & & \downarrow j_* & & \downarrow j'_* & & \downarrow j_* \\ 0 & \longrightarrow & GW^{n-1}(\mathbb{A}^n) & \longrightarrow & GW^{n-1}(\mathbb{A}^n - 0) & \xrightarrow{\partial} & W_{\{0\}}^n(\mathbb{A}^n) \longrightarrow 0 \end{array}$$

where the lines are the exact sequence of localization obtained in the proof of the above lemma. By dévissage, the homomorphism j_* on the right is an isomorphism. \square

3. THE SYMBOL Φ

Let R be a k -algebra. Any unimodular element (a_1, \dots, a_n) yields a morphism $f : \text{Spec}(R) \rightarrow (\mathbb{A}^n - 0)$. Using the above section, we get a map

$$\phi : Um_n(R) \rightarrow GW_{red}^{n-1}(R)$$

defined by $\phi(f) = f^*(\theta(x_1, \dots, x_n))$.

Lemma 3.1. *Suppose that R is regular. Then the map $\phi : Um_n(R) \rightarrow GW_{red}^{n-1}(R)$ is a Steinberg symbol.*

Proof. Let v and w be unimodular rows. We say that v and w are elementarily homotopic if there is a morphism $F : \text{Spec}(R) \times \mathbb{A}^1 \rightarrow (\mathbb{A}^n - 0)$ such that $F(0) = v$ and $F(1) = w$. Since $GW_{red}^{n-1}(R)$ is homotopy invariant, it is clear that $\phi(v) = \phi(w)$ if v and w are elementarily homotopic. We conclude from [12, Theorem 2.1] that ϕ induces a map

$$\phi : Um_n(R)/E_n(R) \rightarrow GW_{red}^{n-1}(R).$$

We next prove that relation (2) holds.

Let $a = (a_1, \dots, a_{n-1}) \in R^{n-1}$ be such that (a, x) and $(a, 1-x)$ are unimodular. Observe that the endomorphism α of $Kos(a) \oplus Kos(a)$ given by the matrix

$$\begin{pmatrix} 1 & x-1 \\ 1 & x \end{pmatrix}$$

is an automorphism. A straightforward computation shows that

$$(T^{n-1}\alpha^\vee)(x\rho(a) \perp (1-x)\rho(a))\alpha = \rho(a) \perp x(1-x)\rho(a)$$

where \perp denotes the orthogonal sum. It follows that

$$x\rho(a) + (1-x)\rho(a) = \rho(a) + x(1-x)\rho(a)$$

in $GW_{red}^{n-1}(R)$. But $\rho(a) = \phi(a, 1) = 0$ (observe that $\rho(a) = 0$ in $GW_{red}^{n-1}(R)$, but that it is in general not trivial in $GW_{V(a)}^{n-1}(R)$). \square

In particular, ϕ induces a map $\phi : Um_n(R)/E_n(R) \rightarrow GW_{red}^{n-1}(R)$. In case n is odd, we next show that ϕ factors through the action of $SL_n(R)$.

Proposition 3.2. *Let R be a regular k -algebra and n be an odd integer. Then $\phi(a) = e(P(a), \chi(a))$ for any $a \in Um_n(R)$.*

Proof. Let $a \in Um_n(R)$ and $P(a)$ its associated stably free module. Consider the section $s(a) : P(a) \rightarrow R$ defined in Section 1. Let $Kos(s(a))$ be the Koszul complex associated to this section and

$$\rho(a) : Kos(s(a)) \rightarrow T^{n-1}Kos(s(a))^\vee$$

the symmetric isomorphism defining the Euler class.

Let $u : R^{n-1} \rightarrow P(a)$ defined by $u(e_{2j}) = f_{2j-1}$, $u(e_{2j-1}) = -f_{2j}$ for any $1 \leq j \leq (n-1)/2$. Taking exterior powers, we get homomorphisms

$$\wedge^i u : \wedge^i(R^{n-1}) \rightarrow \wedge^i(P(a))$$

and it is not hard to check that this induces a morphism of complexes

$$U : Kos(a_1, \dots, a_{n-1}) \rightarrow Kos(s(a)).$$

The homology of both these complexes are concentrated on the closed subset $V(a_1, \dots, a_{n-1})$ of $\text{Spec}(R)$. Therefore U is a quasi-isomorphism on the open complement of $V(a_1, \dots, a_{n-1})$. Since a is unimodular, $V(a_1, \dots, a_{n-1})$ is included in the principal open subset $D(a_n)$ of $\text{Spec}(R)$. But u is an isomorphism on $D(a_n)$ and it follows that U is a quasi-isomorphism.

We now prove that U is in fact an isometry between $\rho(a)$ and $\theta(a_1, \dots, a_n)$. To do this, consider the following diagram

$$\begin{array}{ccc} \wedge^i R^{n-1} & \xrightarrow{\wedge^i u} & \wedge^i P(a) \\ \downarrow & & \downarrow \varphi_i \\ (\wedge^{n-1-i} R^{n-1})^\vee & \xleftarrow{(\wedge^{n-1-i} u)^\vee} & (\wedge^{n-1-i} P(a))^\vee \end{array}$$

Since $-f_{2j} \wedge f_{2j-1} = f_{2j-1} \wedge f_{2j}$, we get

$$(\wedge^{n-1-i} u)^\vee \varphi_i \wedge^i u (e_{m_1} \wedge \dots \wedge e_{m_i})(e_{r_1} \wedge \dots \wedge e_{r_{n-1-i}}) = (-1)^{\text{sign}(\sigma)} \chi(a) (f_1 \wedge \dots \wedge f_{n-1})$$

if σ is a permutation such that $\sigma(m_1, \dots, m_i, r_1, \dots, r_{n-1-i}) = (1, \dots, n-1)$ and

$$(\wedge^{n-1-i} u)^\vee \varphi_i \wedge^i u (e_{m_1} \wedge \dots \wedge e_{m_i})(e_{r_1} \wedge \dots \wedge e_{r_{n-1-i}}) = 0$$

if $e_{m_1} \wedge \dots \wedge e_{m_i} \wedge e_{r_1} \wedge \dots \wedge e_{r_{n-1-i}} = 0$. It follows that U induces an isometry between $\rho(a)$ and $\theta(a_1, \dots, a_n)$. Whence the result. \square

Corollary 3.3. *Let R be a regular k -algebra and n be an odd integer. The map $\phi : Um_n(R) \rightarrow GW_{red}^{n-1}(R)$ induces a map*

$$\Phi : Um_n(R)/SL_n(R) \rightarrow GW_{red}^{n-1}(R).$$

Proof. The Euler class is invariant under the action of $SL(P(a))$. \square

Remark 3.4. If R is regular of Krull dimension 2 then $Um_3(R)/SL_3(R)$ has an abelian group structure such that Φ is an injective homomorphism, with cokernel the Chow group $CH^2(X)$ ([6, Theorem 7.3]).

Remark 3.5. In case n is even, then ϕ doesn't factor through the action of SL_n as easily seen by considering $n = 2$.

Consider the affine scheme $S_n = \text{Spec}(k[x_1, \dots, x_n, y_1, \dots, y_n]/(\sum x_i y_i - 1))$. If R is any k -algebra, $a = (a_1, \dots, a_n)$ is a unimodular row and $b = (b_1, \dots, b_n)$ is such that $ba^t = 1$, then we obtain a morphism $\text{Spec}(R) \rightarrow S_n$ mapping x_i to a_i and y_i to b_i . Conversely, any morphism $\text{Spec}(R) \rightarrow S_n$ yields such an a and b . We therefore call S_n the *unimodular affine scheme*.

The ring homomorphism $k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n, y_1, \dots, y_n]/(\sum x_i y_i - 1)$ given by $x_i \mapsto x_i, y \mapsto y_1$ for any $1 \leq i \leq n$ induces a morphism $p_n : S_n \rightarrow (\mathbb{A}^n - 0)$, which is easily checked to have affine fibres. It follows then from the Mayer-Vietoris sequence ([16, Theorem 1]) that $p_n^* : GW^{n-1}(\mathbb{A}^n - 0) \rightarrow GW^{n-1}(S_n)$ is an isomorphism, showing that from a cohomological viewpoint the choice of a section of a unimodular row is not important.

Proposition 3.6. *If $n \geq 3$ is odd, the map $p_n : S_n \rightarrow (\mathbb{A}^n - 0)$ induces a surjective map*

$$\Phi : Um_n(S_n)/SL_n(S_n) \rightarrow W(k)$$

with $\Phi(x_1, \dots, x_n) = \langle 1 \rangle$.

Proof. In view of Lemma 2.4 and Proposition 3.3, it remains only to show that Φ is surjective. It is obviously sufficient to prove that $\phi : Um_n(S_n)/E_n(S_n) \rightarrow W(k)$ is surjective to conclude.

Suppose that $n > 3$. Setting $x_i = 0$ and $y_i = 0$ for $i > 3$, we get closed immersions $j' : (\mathbb{A}^3 - 0) \rightarrow (\mathbb{A}^n - 0)$ and $u : S_3 \rightarrow S_n$ such that the diagram

$$\begin{array}{ccc} S_3 & \xrightarrow{u} & S_n \\ p_3 \downarrow & & \downarrow p_n \\ (\mathbb{A}^3 - 0) & \xrightarrow{j'} & (\mathbb{A}^n - 0) \end{array}$$

commutes. Now we can define a map

$$\beta : Um_3(S_3)/E_3(S_3) \rightarrow Um_n(S_n)/E_n(S_n)$$

by $\beta(\bar{a}_1, \bar{a}_2, \bar{a}_3) = (a_1, a_2, a_3, x_4, \dots, x_n)$, where a_i are any lifts of \bar{a}_i . Seeing p_3 and p_n as unimodular rows, it is easily checked that $\beta(p_3) = p_n$. It follows that the following diagram

$$\begin{array}{ccc} Um_3(S_3)/E_3(S_3) & \xrightarrow{\beta} & Um_n(S_n)/E_n(S_n) \\ \phi \downarrow & & \downarrow \phi \\ GW_{red}^2(S_3) & \xrightarrow{u_*} & GW_{red}^{n-1}(S_n) \end{array}$$

commutes. Since u_* is an isomorphism by Lemma 2.7, it suffices to check that $\phi : Um_3(S_3)/E_3(S_3) \rightarrow W(k)$ is surjective to conclude.

Recall that $\tilde{K}_0 Sp(S_3)$ is defined by the exact sequence

$$0 \longrightarrow K_0(k) \xrightarrow{p^* H} GW^2(S_3) \longrightarrow \tilde{K}_0 Sp(S_3) \longrightarrow 0$$

where $H : K_0(k) \rightarrow GW^2(k)$ is the hyperbolic functor (see [11, §2.1] for instance) and $p^* : GW^2(k) \rightarrow GW^2(S_3)$ is the pull-back homomorphism. Since H is an isomorphism, it follows that there is a natural isomorphism $\tilde{K}_0 Sp(S_3) \rightarrow GW_{red}^2(S_3)$. Under this identification, the symbol ϕ coincides with the Vaserstein symbol

$$V : Um_3(S_3)/E_3(S_3) \rightarrow \tilde{K}_0 Sp(S_3)$$

defined in [21, Theorem 5.2] (see also [5, §5]). In particular, this shows that ϕ satisfies the Vaserstein rule: For any unimodular rows (a_1, a_2, a_3) and (a_1, b_2, b_3) , we have

$$\phi(a_1, a_2, a_3) + \phi(a_1, b_2, b_3) = \phi(a_1, (a_2 \quad a_3) \cdot \begin{pmatrix} b_2 & b_3 \\ -c_3 & c_2 \end{pmatrix})$$

where c_2, c_3 are such that $b_2 c_2 + b_3 c_3 \equiv 1 \pmod{a_1}$. This proves that for any $\alpha_1, \dots, \alpha_n \in k^\times$, there exists $b, c \in k[x_1, x_2, x_3, y_1, y_2, y_3]/(\sum x_i y_i - 1)$ such that

$$\phi(x_1, x_2, \alpha_1 x_3) + \dots + \phi(x_1, x_2, \alpha_n x_3) = \phi(x_1, b, c).$$

Remark 2.6 then shows that $\langle \alpha_1, \dots, \alpha_n \rangle$ is in the image of ϕ . □

Remark 3.7. If $n \geq 5$ is an odd integer, the map Φ is not injective. Indeed, the proof of the above proposition shows that Φ satisfies the Vaserstein rule. It follows from [19, lemma 3.5 (v)] that $\Phi(a_1, a_2, \dots, a_n) + \Phi(b_1^2, a_2, \dots, a_n) = \Phi(a_1 b_1^2, a_2, \dots, a_n)$

for any unimodular rows (a_1, a_2, \dots, a_n) and (b_1, a_2, \dots, a_n) . Using this and [19, Lemma 3.5 (iii)], we see that

$$\Phi(x_1^2, x_2, \dots, x_n) = \Phi(x_1, x_2, \dots, x_n) + \Phi(-x_1, x_2, \dots, x_n)$$

By Remark 2.6, this reads as

$$\Phi(x_1^2, x_2, \dots, x_n) = \langle 1, -1 \rangle \Phi(x_1, x_2, \dots, x_n) = 0$$

But it is well-known that (x_1^2, x_2, \dots, x_n) is not completable ([17]). Hence Φ is not injective.

Question 3.8. *Is the map $\Phi : Um_3(S_3)/SL_3(S_3) \rightarrow W(k)$ bijective?*

4. THE DEGREE MAP

4.1. The map. Let $n \geq 2$ and $g : (\mathbb{A}^n - 0) \rightarrow (\mathbb{A}^n - 0)$ be a morphism. Taking global sections, we see that there is a commutative diagram

$$\begin{array}{ccc} (\mathbb{A}^n - 0) & \xrightarrow{g} & (\mathbb{A}^n - 0) \\ i \downarrow & & \downarrow i \\ \mathbb{A}^n & \xrightarrow{g'} & \mathbb{A}^n \end{array}$$

where $i : (\mathbb{A}^n - 0) \rightarrow \mathbb{A}^n$ is the inclusion, and $g' : \mathbb{A}^n \rightarrow \mathbb{A}^n$ is the morphism associated to some $\varphi : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ with $\varphi(\mathfrak{m}) \subset \mathfrak{m}$. We then get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & GW^{n-1}(\mathbb{A}^n) & \xrightarrow{i^*} & GW^{n-1}(\mathbb{A}^n - 0) & \xrightarrow{\partial} & W_{\{0\}}^n(\mathbb{A}^{n-1}) \longrightarrow 0 \\ & & \downarrow (g')^* & & \downarrow g^* & & \downarrow \\ 0 & \longrightarrow & GW^{n-1}(\mathbb{A}^n) & \xrightarrow{i^*} & GW^{n-1}(\mathbb{A}^n - 0) & \xrightarrow{\partial} & W_{\{0\}}^n(\mathbb{A}^n) \longrightarrow 0 \end{array}$$

Lemma 4.1. *We have $g^*(\theta(x_1, \dots, x_n)) = \rho(\varphi(x_1), \dots, \varphi(x_n)) \cdot \theta(x_1, \dots, x_n)$ in $GW_{red}^{n-1}(\mathbb{A}^n - 0)$.*

Proof. It suffices to compute the element $\partial g^*(\theta(x_1, \dots, x_n))$ in $W_{\{0\}}^n(\mathbb{A}^n) = W(k)$. This is exactly $\rho(\varphi(x_1), \dots, \varphi(x_n))$. \square

Remark 4.2. Here, we see $\rho(\varphi(x_1), \dots, \varphi(x_n))$ as an element of $W(k)$ using dévissage (see Section 2.3).

Definition 4.3. We call the class of $\rho(\varphi(x_1), \dots, \varphi(x_n))$ in $W(k)$ the *degree* of g and we denote it by $\deg(g)$.

Remark 4.4. This terminology is justified as follows: Suppose that k admits a real embedding $k \subset \mathbb{R}$. Pulling back, we get a morphism $g : (\mathbb{A}_{\mathbb{R}}^n - 0) \rightarrow (\mathbb{A}_{\mathbb{R}}^n - 0)$ and, taking the rational points, a C^∞ map $g : (\mathbb{R}^n - 0) \rightarrow (\mathbb{R}^n - 0)$. Now one can associate to any C^∞ map $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ a topological degree of f , denoted by $\deg_{\text{top}}(f)$, as the sum of the signs of the Jacobian of f at all preimages (under f) near 0 of a regular value of f near 0. In [8], the authors show that this degree can be computed as the degree (in the sense of Definition 4.3) of a polynomial map f' approximating f . Thus, our degree and the topological degree coincide in that case.

4.2. The counter-example. Let $n \geq 2$ and $g : (\mathbb{A}^n - 0) \rightarrow (\mathbb{A}^n - 0)$ be the morphism defined by the ring homomorphism $\varphi : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ with $\varphi(x_1) = x_1^2 - x_2^2$, $\varphi(x_2) = x_1x_2$ and $\varphi(x_i) = x_i$ for $3 \leq i \leq n$. Observe that $k[x_1, \dots, x_n]/(\varphi(x_1), \dots, \varphi(x_n))$ is a finite length module of length 4. Following Section 2.3, it suffices to compute the class in $W^{fl}(k[x_1, \dots, x_n]_{\mathfrak{m}})$ of this module endowed with the form

$$\rho : k[x_1, \dots, x_n]/(\varphi(x_1), \dots, \varphi(x_n)) \rightarrow k[x_1, \dots, x_n]/(\widehat{(\varphi(x_1), \dots, \varphi(x_n))})$$

defined by $\rho(1) = \text{Kos}(\varphi(x_1), \dots, \varphi(x_n))$.

Lemma 4.5. *We have $\deg(g) = \langle 1, 1 \rangle$.*

Proof. Let $R = k[x, y]$ and $\mathfrak{m} = (x, y)$. By dévissage, it suffices clearly to prove that $\deg(g) = \langle 1, 1 \rangle$ if $g : (\mathbb{A}^2 - 0) \rightarrow (\mathbb{A}^2 - 0)$ is the map induced by $\varphi : R \rightarrow R$ with $\varphi(x) = x^2 - y^2$ and $\varphi(y) = xy$. We denote by M the ideal generated by the sequence $(x^2 - y^2, xy)$ and by $\rho : R/M \rightarrow \widehat{R/M}$ the symmetric isomorphism $\rho(\varphi(x), \varphi(y))$. Consider the class of $x^2 \in R/M$. A direct computation shows that it is non trivial and annihilated by \mathfrak{m} . Moreover, $R/(M, x^2) = R/\mathfrak{m}^2$ and we then get the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R/\mathfrak{m} & \xrightarrow{x^2} & R/M & \xrightarrow{\pi} & R/\mathfrak{m}^2 \longrightarrow 0 \\ & & & & \downarrow \rho & & \\ 0 & \longrightarrow & \widehat{R/\mathfrak{m}^2} & \xrightarrow{\widehat{\pi}} & \widehat{R/M} & \xrightarrow{x^2} & \widehat{R/\mathfrak{m}} \longrightarrow 0 \end{array}$$

We next check that $\widehat{x^2}(\text{Kos}(x^2 - y^2, xy)) = \text{Kos}(x, y)$. This is clear from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} & R^2 & \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} & R & \longrightarrow & R/\mathfrak{m} \longrightarrow 0 \\ & & \parallel & & \downarrow \begin{pmatrix} x & 0 \\ y & x \end{pmatrix} & & \downarrow x^2 & & \downarrow x^2 \\ 0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} -xy \\ x^2 - y^2 \end{pmatrix}} & R^2 & \xrightarrow{\begin{pmatrix} x^2 - y^2 & xy \end{pmatrix}} & R & \longrightarrow & R/\mathfrak{m} \longrightarrow 0. \end{array}$$

It follows then that $\widehat{x^2}\rho x^2 = 0$, proving that $x^2 : R/\mathfrak{m} \rightarrow R/M$ is a sub-Lagrangian of $\rho : R/M \rightarrow \widehat{R/M}$. There exists then a homomorphism $\alpha : R/\mathfrak{m}^2 \rightarrow \widehat{R/\mathfrak{m}}$ such

that the diagram commutes:

$$\begin{array}{ccccccc}
 & & & & \mathfrak{m}/\mathfrak{m}^2 & & \\
 & & & & \downarrow i & & \\
 0 & \longrightarrow & R/\mathfrak{m} & \xrightarrow{x^2} & R/M & \xrightarrow{\pi} & R/\mathfrak{m}^2 \longrightarrow 0 \\
 & & \downarrow \widehat{\alpha} & & \downarrow \rho & & \downarrow \alpha \\
 0 & \longrightarrow & \widehat{R/\mathfrak{m}^2} & \xrightarrow{\widehat{\pi}} & \widehat{R/M} & \xrightarrow{\widehat{x^2}} & \widehat{R/\mathfrak{m}} \longrightarrow 0 \\
 & & \downarrow \widehat{i} & & & & \\
 & & \widehat{\mathfrak{m}/\mathfrak{m}^2} & & & &
 \end{array}$$

By commutativity of the diagram, $\alpha(1) = \text{Kos}(x, y)$ and its kernel is $\mathfrak{m}/\mathfrak{m}^2$. The snake Lemma yields an isomorphism $\psi : \mathfrak{m}/\mathfrak{m}^2 \rightarrow \widehat{\mathfrak{m}/\mathfrak{m}^2}$ which is easily checked to be symmetric. The sub-Lagrangian reduction ([2, §1.1.5]) shows then that $\rho = \psi$ in $W^{fl}(R)$. In order to compute ψ , we need a better understanding of the groups and homomorphisms in the diagram. We first compute $\widehat{R/\mathfrak{m}^2}$.

The projective resolution

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ -x & -y \\ 0 & x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} R \longrightarrow R/\mathfrak{m}^2 \longrightarrow 0$$

shows that $\widehat{R/\mathfrak{m}^2}$ is generated by two extensions, the push-out by the homomorphisms $\begin{pmatrix} 1 & 0 \end{pmatrix} : R^2 \rightarrow R$ and $\begin{pmatrix} 0 & 1 \end{pmatrix} : R^2 \rightarrow R$. We will denote by ϵ_1 the first extension and by ϵ_2 the second extension. Now the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} -xy \\ x^2 - y^2 \end{pmatrix}} & R^2 & \xrightarrow{\begin{pmatrix} x^2 - y^2 & xy \end{pmatrix}} & R \longrightarrow R/M \longrightarrow 0 \\
 & & \downarrow -x & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} & & \parallel \\
 0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}} & R^3/(0, -y, x) & \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} & R \longrightarrow R/\mathfrak{m}^2 \longrightarrow 0
 \end{array}$$

shows that $\widehat{\pi}(\epsilon_1) = -x\text{Kos}(x^2 - y^2, xy)$. A straightforward computation shows that $\widehat{\pi}(\epsilon_2) = y\text{Kos}(x^2 - y^2, xy)$.

Now $\mathfrak{m}/\mathfrak{m}^2$ is generated by the classes of x and y , yielding an isomorphism $(R/\mathfrak{m})^2 \simeq \mathfrak{m}/\mathfrak{m}^2$. We get a projective resolution

$$0 \longrightarrow R^2 \xrightarrow{\begin{pmatrix} -y & 0 \\ x & 0 \\ 0 & -y \\ 0 & x \end{pmatrix}} R^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} \mathfrak{m}/\mathfrak{m}^2 \longrightarrow 0$$

and the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^2 & \xrightarrow{\begin{pmatrix} -y & 0 \\ x & 0 \\ 0 & -y \\ 0 & x \end{pmatrix}} & R^4 & \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} & R^2 & \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} & \mathfrak{m}/\mathfrak{m}^2 & \longrightarrow & 0 \\
& & \downarrow \begin{pmatrix} -1 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} x & y \end{pmatrix} & & \downarrow i & & \\
0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}} & R^3/(0, -y, x) & \xrightarrow{\begin{pmatrix} x^2 & xy & y^2 \end{pmatrix}} & R & \longrightarrow & R/\mathfrak{m}^2 & \longrightarrow & 0
\end{array}$$

shows that $\widehat{i}(\epsilon_1) = \begin{pmatrix} -1 & 0 \end{pmatrix}$. Similarly, $\widehat{i}(\epsilon_2) = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

Using our computations, we see that $\psi : \mathfrak{m}/\mathfrak{m}^2 \rightarrow \widehat{\mathfrak{m}/\mathfrak{m}^2}$ is given (in the basis x, y) by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $\deg(g) = \langle 1, 1 \rangle$. \square

Corollary 4.6. *Let k be a field such that $\sqrt{-1} \notin k$ and let $n \geq 3$ be an odd integer. Consider the unimodular row $(x_1, \dots, x_n) \in Um_n(S_n)$ and the map $g : (\mathbb{A}^n - 0) \rightarrow (\mathbb{A}^n - 0)$ defined by the algebra homomorphism $\varphi : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ given by $\varphi(x_1) = x_1^2 - x_2^2$, $\varphi(x_2) = x_1x_2$ and $\varphi(x_i) = x_i$ for $3 \leq i \leq n$. Then the unimodular row $(x_1^2 - x_2^2, x_1x_2, x_3, \dots, x_n) \in Um_n(S_n)$ is not completable in an invertible matrix.*

Proof. Consider the following diagram:

$$S_n \xrightarrow{p} (\mathbb{A}^n - 0) \xrightarrow{g} (\mathbb{A}^n - 0)$$

Then $(x_1^2 - x_2^2, x_1x_2, x_3, \dots, x_n)$ is the unimodular row associated to the morphism of schemes $gp : S_3 \rightarrow \mathbb{A}^3 - 0$. We have

$$\phi(gp) = (gp)^*(\theta(x_1, \dots, x_n)) = p^*g^*(\theta(x_1, \dots, x_n)).$$

By definition, $g^*(\theta(x_1, \dots, x_n)) = \deg(g)$, which is $\langle 1, 1 \rangle$ by Lemma 4.5. Then $p^*g^*(\theta(x_1, \dots, x_n)) = \langle 1, 1 \rangle \cdot \theta(x_1, \dots, x_n)$ and $\Phi(x_1^2 - x_2^2, x_1x_2, x_3, \dots, x_n) = \langle 1, 1 \rangle$. Now $\langle 1, 1 \rangle = 0$ in $W(k)$ if and only if -1 is a square in k . It follows that the unimodular row $(x_1^2 - x_2^2, x_1x_2, x_3, \dots, x_n)$ is not completable. \square

We now give the counter-example to M. V. Nori's question.

Theorem 4.7. *Let k be a field such that $\sqrt{-1} \notin k$. Consider the unimodular row $(x_1, x_2, x_3) \in Um_3(S_3)$ and the map $g : (\mathbb{A}^3 - 0) \rightarrow (\mathbb{A}^3 - 0)$ defined by the algebra homomorphism $\varphi : k[x_1, x_2, x_3] \rightarrow k[x_1, x_2, x_3]$ given by $\varphi(x_1) = x_1^2 - x_2^2$, $\varphi(x_2) = x_1x_2$ and $\varphi(x_3) = x_3$. Then $k[x_1, x_2, x_3]/(x_1^2 - x_2^2, x_1x_2, x_3)$ is of length 4, but $(x_1^2 - x_2^2, x_1x_2, x_3) \in Um_3(S_3)$ is not completable.*

Proof. Clear from the above corollary. \square

This example shows that Nori's question has to be strengthened. We propose the following:

Question 4.8 (Strong Nori's question). *Let R be a k -algebra, $n \in \mathbb{N}$ be odd and let $f : \text{Spec}(R) \rightarrow (\mathbb{A}^n - 0)$ be a unimodular row. Let $\varphi : k[x_1, \dots, x_n] \rightarrow k[x_1, \dots, x_n]$ such that $\varphi(\mathfrak{m}) \subset \mathfrak{m}$ and that $(n-1)!$ divides the length of $k[x_1, \dots, x_n]/\varphi(\mathfrak{m})$. Let $g : (\mathbb{A}^n - 0) \rightarrow (\mathbb{A}^n - 0)$ the morphism induced by φ . If the degree $\deg(g) = 0$, then the unimodular row $gf : \text{Spec}(R) \rightarrow (\mathbb{A}^n - 0)$ is completable.*

Theorem 4.9. *If $\Phi : \text{Um}_3(S_3)/SL_3(S_3) \rightarrow W(k)$ is injective, then Question 4.8 has an affirmative answer for $n = 3$.*

Proof. Let $f : \text{Spec}(R) \rightarrow (\mathbb{A}^3 - 0)$ be a unimodular row. Then f factors through S_3 as indicated in the following diagram:

$$\begin{array}{ccc} \text{Spec}(R) & \xrightarrow{f'} & S_3 \\ & \searrow f & \downarrow p \\ & & (\mathbb{A}^3 - 0) \end{array}$$

Let $g : (\mathbb{A}^3 - 0) \rightarrow (\mathbb{A}^3 - 0)$ be a morphism as in the conjecture. Consider the unimodular row $gp : S_3 \rightarrow (\mathbb{A}^3 - 0)$. Let $r : SL_3 \rightarrow (\mathbb{A}^3 - 0)$ be the projection to the first row. If $\deg(g) = 0$ and Φ is injective, it follows that $gp : S_3 \rightarrow (\mathbb{A}^3 - 0)$ factors through SL_3 . We then have a commutative diagram:

$$\begin{array}{ccccc} & & S_3 & & \\ & \nearrow f' & \downarrow p & \searrow \exists q & \\ \text{Spec}(R) & \xrightarrow{f} & (\mathbb{A}^3 - 0) & & SL_3 \\ & \searrow gf & \downarrow g & \swarrow r & \\ & & (\mathbb{A}^3 - 0) & & \end{array}$$

Therefore $gf = rqf'$ factors through SL_3 , thus showing that gf is completable. \square

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